Math 72 8.1 Quadratic Functions and Their Graphs

Objectives

1) Graph a quadratic function
2) Use vertex formula to find vertex of a quadratic function
3) Find the maximum or minimum value of a quadratic function

8.2 Transformations and Translations of Parabolas

Objectives

1) Graph parabolas when $|a| \neq 1$ $f(x) = ax^2$
2) Use vertical translations to graph parabolas.
$f(x) = ax^2 + k$
3) Use horizontal translations to graph parabolas
$f(x) = a(x-h)^2$
4) Use vertex form of equation of parabola to find vertex
$f(x) = a(x-h)^2 + k$
5) Find vertex form using vertex formula
6) Find vertex form using completing the square.
   - when $a = 1$
   - when $a \neq 1$.

A quadratic function is a degree 2 polynomial

$f(x) = ax^2 + bx + c$

Its shape is called a parabola

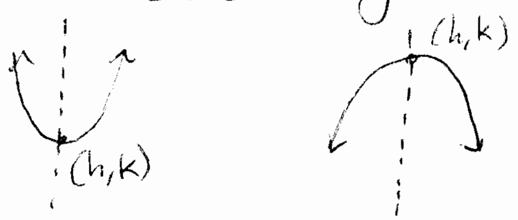
If $a > 0$ the parabola opens upward

If $a < 0$ the parabola opens downward

a is called the leading coefficient.

The left and right sides of a parabola are mirror images of each other — this is called symmetry.

The line which divides the parabola in half is called the axis of symmetry.



MathXL
dotted lines when graphing axis of symmetry

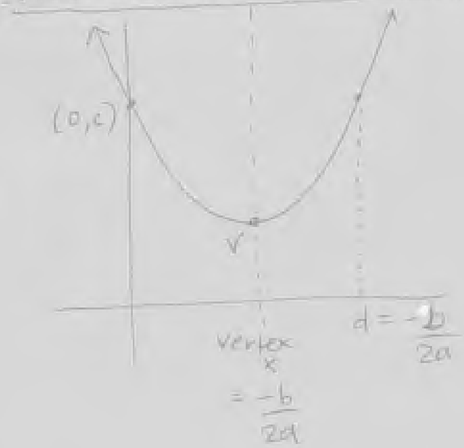
The lowest point (of an upward parabola) or the highest point (of a downward parabola) is called the vertex. (h, k).

The equation of the axis of symmetry is $x = h$, the x-coordinate of the vertex.

The vertex formula $h = \dfrac{-b}{2a}$ gives the x-coordinate.

$k = f(h)$

Rockswold's Derivation of the Vertex Formula



Ⓐ Find the y-intercept of a quadratic function $f(x) = ax^2 + bx + c$

y-intercept: point on the graph where graph crosses the y-axis.
set $x = 0$.

$$f(0) = a(0)^2 + b(0) + c = 0 + 0 + c = c$$

$\boxed{\text{y int } (0, c)}$

Ⓑ Find the x-value $x = d$ so that $f(d) = c$, the same y-coordinate as the y-intercept.

$f(x) = c$ — quadratic equation set $= 0$

$$ax^2 + bx + c = c$$
$$\underline{-c} \quad \underline{-c}$$

$ax^2 + bx = 0$ — factor GCF x

$x(ax + b) = 0$ — set each factor $= 0$

$\underbrace{x = 0}_{\text{y-int}}$ $\quad ax + b = 0$

$ax = -b$

$\underbrace{x = \frac{-b}{a}}_{\text{the point}}$ $\boxed{x = d = \frac{-b}{a}}$

Ⓒ The x-coordinate of the vertex is the midpoint between the y-intercept and the point found in Ⓑ.

Find the midpoint between $x = 0$ and $x = \frac{-b}{a}$

$$x = \frac{0 + \frac{-b}{a}}{2}$$

$$= \frac{-b}{a} \div 2$$

$$= \frac{-b}{a} \cdot \frac{1}{2}$$

$\boxed{x = \frac{-b}{2a}}$ ← This is the vertex formula for the x-coordinate of the vertex quadratic function.

Sketch graph

No ① $f(x) = x^2 - 1$

x	$f(x)$
-3	8
-2	3
-1	0
Vertex → 0	-1
1	0
2	3
3	8

symmetry

Find vertex using vertex formula

$h = \dfrac{-b}{2a} \Rightarrow$ $\qquad f(x) = 1x^2 + 0x - 1$

$a=1 \quad b=0 \quad c=-1$

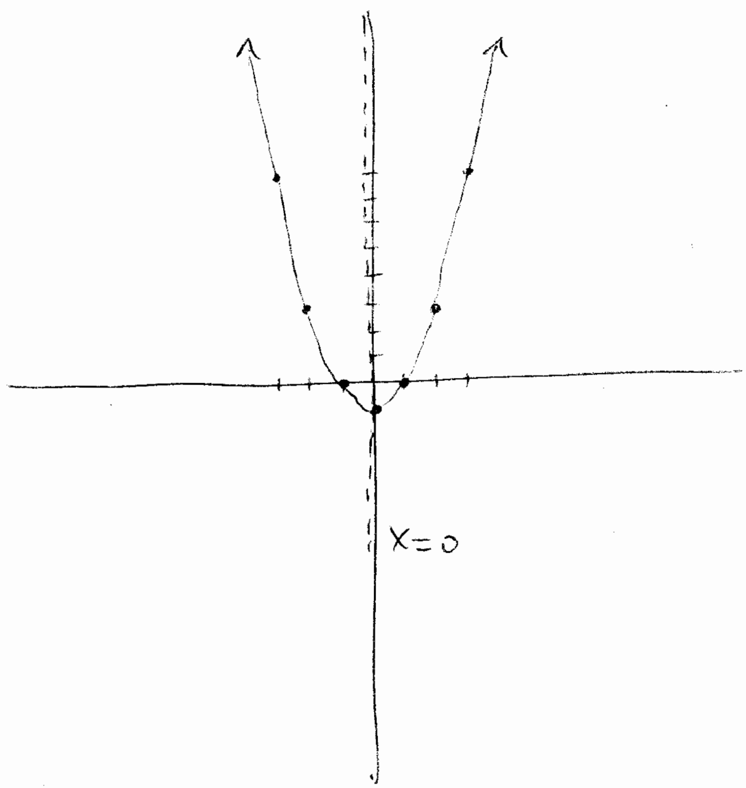
$h = \dfrac{-(0)}{2(1)} = 0$

$k = f(0)$

$= (0)^2 - 1$ $\qquad$ subst

$= -1$

Vertex $(0, -1)$



axis of symmetry
$x = 0$

(2) Sketch graph + axis of symmetry

$f(x) = x^2 + 4x + 1$

vertex $h = \frac{-b}{2a} = \frac{-4}{2(1)} = -2$

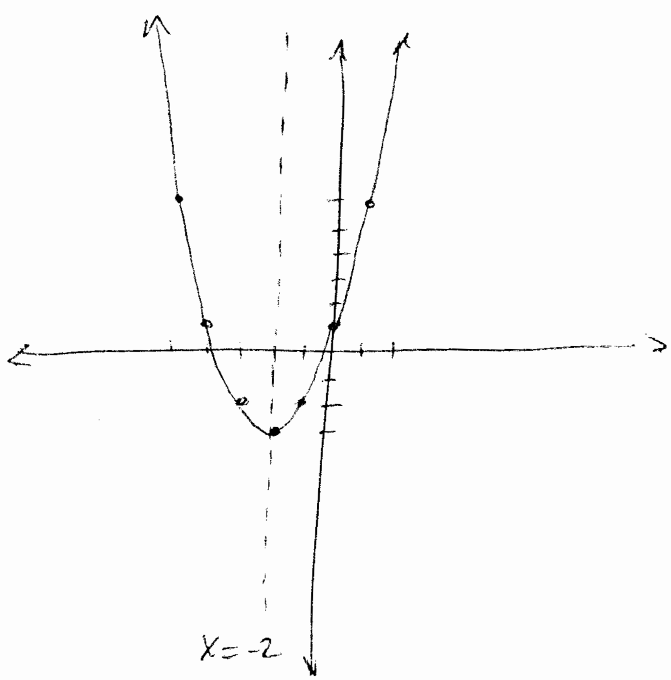
$k = f(-2) = (-2)^2 + 4(-2) + 1$

$= -3$

vertex $(-2, -3)$

	x	$f(x)$
	-5	6
		1
		-2
Vertex →	-2	-3
	-1	-2
	0	1
	1	6
	2	13

symmetry



axis of symmetry

$x = -2$

② $f(x) = x^2 + 4x + 1$

a) Find vertex

b) Sketch graph

c) Graph axis of sym w/ dotted line

d) Find equation of axis of symmetry.

e) Label axis of symmetry with its equation

f) Write $f(x)$ in vertex form.

continued.

f) Vertex form of the equation of a quadratic function

Vertex Form of a Quadratic Function

$$f(x) = a\left(x - \boxed{}\right)^2 + \boxed{}$$

same leading coefficient as $f(x) = ax^2 + bx + c$

subtract (inside parentheses)

substitute x-coordinate of vertex here

add (outside parentheses)

substitute y-coordinate of vertex here

for $f(x) = x^2 + 4x + 1$

$a = 1$

x-coord of vertex $= -2$

y-coord of vertex $= -3$

substitute

$$f(x) = 1\left(x - (-2)\right)^2 + (-3)$$

simplify

$$\boxed{f(x) = (x+2)^2 - 3}$$

check that this is equivalent? FOIL and combine

$$f(x) = (x+2)(x+2) - 3$$

$$= x^2 + 4x + 4 - 3$$

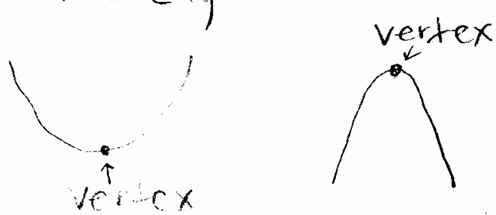
$$f(x) = x^2 + 4x + 1 \checkmark$$

When graphing quadratic functions

1) The shape is a parabola.
This is a smooth, rounded curve without any pointy places.

Yes. No. Yes No

2) The most important point is the vertex.
Always plot the vertex neatly and accurately.



3) Plot at least <u>four</u> additional points to show if the parabola is wider or narrower

Your graph should be accurate to the edges of the grid.

4) A parabola is symmetric: the left side of the graph is the same shape as the right side
Use this to save time and check your work.

When graphing quadratic functions (continued)

5) Draw axes neatly. If scale in x direction or scale in y-direction is <u>not</u> 1, label scales.

6) Place axes so that 5 points will fit on the grid.

7) Extend parabola accurately to edges of grid.

8) Check x-intercepts and y-intercepts are plotted accurately.
(Compare to GC graph if x-ints or y-ints were not requested in your work.)

Remember: To find x-intercepts, set $y=0$ and solve for x.

To find y-intercept, set $x=0$ and solve for y. ($f(x)$)

9) Draw axis of symmetry (if requested) using a dashed line and label it with its equation.

Lessons 45-46

Math 70 11.7 & 11.8 Maximum and Minimum Problems with Quadratic Functions
and GC 32: Use GC to find max/min.

Objectives:

1) Use the vertex formula to find the vertex of a quadratic function $f(x) = ax^2 + bx + c$.
   a. x-coordinate $h = -\dfrac{b}{2a}$, y-coordinate $k = f(h)$
2) Identify whether a quadratic function $f(x) = ax^2 + bx + c$ or $f(x) = a(x-h)^2 + k$ has a maximum or minimum value.
3) Use the vertex formula to find the maximum or minimum value of a quadratic function.
   a. A quadratic function has only one – either a maximum or a minimum, not both.
   b. The max or min occurs at the vertex.

CAUTION: Vocabulary for the maximum/minimum and its location are important because it tells you which coordinate of the vertex is the answer to the question.

   c. The maximum or minimum value is the y-coordinate of the vertex, $y = k = f(h)$, because it is the maximum function value or minimum function value.
   d. The location of the maximum or minimum is the x-coordinate of the vertex. $x = h = -\dfrac{b}{2a}$
4) Use the vertex formula to solve maximum/minimum word problems.

NOTE: The question "find the maximum or minimum" is a general problem. In Math 70, we only do this for quadratic functions, so in Math 70, the key words "Maximum" or "Minimum" connect to "Vertex Formula".

Practice and Examples

yes 1) Does $h(t) = -16t^2 + 10t + 100$ open upward or downward? Does $h(t) = -16t^2 + 10t + 100$ have a maximum or minimum? Why?

yes 2) An object is thrown upward from the top of a 100-ft cliff. Its height h in feet above the base of the cliff after t seconds is given by $h(t) = -16t^2 + 10t + 100$. Find the maximum height of the object and the number of seconds it took to reach that max height.

3) Does $C(x) = 2x^2 + -800x + 92000$ open upward or downward? Does $C(x) = 2x^2 + -800x + 92000$ have a maximum or minimum? Why?

4) The cost C of manufacturing x bicycles at Holladay's Production Plant is given by the function $C(x) = 2x^2 + -800x + 92000$
   a. Find the number of bicycles that must be manufactured to minimize the cost.
   b. Find the minimum cost.

yes 5) The Utah Ski Club sells calendars to raise money. The profit P, in cents, from selling x calendars is given by the equation $P(x) = 360x - x^2$.

a. Find how many calendars must be sold to maximize profit.

b. Find the maximum profit.

6) Find two numbers whose sum is 60 and whose product is as large as possible. [Hint: Let x and 60-x be the two positive numbers. Their product can be described by the function $f(x) = x(60 - x)$.]

yes 7) The length and width of a rectangle must have a sum of 40 cm. Find the dimensions of the rectangle that will have the maximum area.

8) Methane is a gas produced by landfills, natural gas systems, and coal mining that contributes to the greenhouse effect and global warming. Projected methane emissions in the US can be modeled by the quadratic function $f(x) = -0.072x^2 + 1.93x + 173.9$, where f(x) is the amount of methane produced in million metric tons and x is the number of years after 2010.

a. According to this model, what will US emissions of methane be in 2019? (Round to two decimal places.)

b. Will this function have a maximum or a minimum? Explain.

c. In what year will methane emissions in the US be at their maximum/minimum? Round to the nearest whole year.

d. What is the level of methane emissions for that year? (Use your rounded answer from part c.) (Round this answer to 2 decimal places.)

Extras:

9) Find two numbers whose sum is 11 and whose product is a large as possible.

10) Find two numbers whose difference is 10 and whose product is as small as possible.

11) Find two numbers whose difference is 8 and whose product is as small as possible.

12) The length and width of a rectangle must have a sum of 50. Find the dimensions of the rectangle that will have maximum area.

13) Explore finding vertex by GC tables

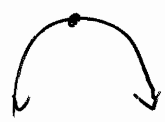
+ Review problems on vertex, x-ints, graphing

① Does $h(t) = -16t^2 + 10t + 100$ open upward or downward?
Does $h(t) = -16t^2 + 10t + 100$ have a maximum or minimum?
Why?

$h(t) = -16t^2 + 10t + 100$

is like $f(x) = -16x^2 + 10x + 100$ where f was changed to h and x was changed to t.

$a = -16$ is negative
so parabola opens $\boxed{\text{downward}}$

A downward parabola  has a $\boxed{\text{maximum value}}$

because the vertex is at the top of all other points (x, y) on the parabola so the y-coordinate of the vertex is greater than all other y-coordinates.

② An object is thrown upward from the top of a 100-ft cliff. Its height h in feet above the base of the cliff after t seconds is given by $h(t) = -16t^2 + 10t + 100$. Find the maximum height of the object and the number of seconds it took to reach that max height.

maximum occurs at the vertex.

If the function is written $f(x) = a(x-h)^2 + k$, we can read the vertex from the function.

But if the function is written $f(x) = ax^2 + bx + c$, we find the vertex using the vertex formula.

Vertex Formula for $f(x) = ax^2 + bx + c$
x-coordinate of vertex $= \dfrac{-b}{2a}$
y-coordinate of vertex $= f\left(\dfrac{-b}{2a}\right)$

* Memorize.

↑ Want to know where this came from?
Use completing-the-square... to see this done, look at the end of this lesson.

Math 70

Find the vertex of $h(t) = -16t^2 + 10t + 100$

x-coord $\approx$ t coordinate $= \frac{-b}{2a} = \frac{-10}{2(-16)} = \frac{5}{16} = .3125$

y-coord $\approx$ h coord $= h\left(\frac{5}{16}\right) = -16\left(\frac{5}{16}\right)^2 + 10\left(\frac{5}{16}\right) + 100$
$f\left(-\frac{b}{2a}\right)$ $\quad h\left(-\frac{b}{2a}\right)$

$= \frac{1625}{16} = 101.5625$

The vertex is $\left(\frac{5}{16}, \frac{1625}{16}\right)$ or $(.3125, 101.5625)$.

The maximum height is the h (or y) coordinate of the vertex.

$\boxed{\text{max height } 101.5625 \text{ ft}}$

The time the object reaches the max height is the t (or x) coordinate of the vertex.

$\boxed{\text{time of max height } 0.3125 \text{ sec}}$.

③ Does $C(x) = 2x^2 - 800x + 92000$ open upward or down? Does it have a max or min? Why?

$C(x) = ax^2 + bx + c$
$\uparrow$
$2x^2 - 800x + 92000$

$a = 2$ is positive

$\boxed{\begin{array}{l} a > 0 \\ \text{opens up} \\ \text{has a minimum} \end{array}}$

④ The cost C of manufacturing x bicycles is given by $C(x) = 2x^2 - 800x + 92000$.

a) Find the number of bicycles to minimize cost.

b) Find minimum cost.

These key words, in Math 70, mean "find the vertex".

Math 70

vertex formula: $x = \frac{-b}{2a}$

$= \frac{-(-800)}{2(2)}$

$= \frac{800}{4}$

x coordinate of vertex $= \boxed{200 \text{ bicycles}}$ a)

y-coordinate of vertex $= C(200)$

$= 2(200)^2 - 800(200) + 92000$

$= \boxed{\$12000 \text{ minimum cost}}$ b)

(5) Profit P, in cents, from selling x calendars is $P(x) = 360x - x^2$.

a) How many calendars to maximize profit?

b) Find max profit.

P(x) is not written in standard form!

$P(x) = -x^2 + 360x \implies a = -1$ neg. ∩

$b = 360$

$c = 0$.

vertex formula

x-coordinate of vertex $= \frac{-b}{2a} = \frac{-360}{2(-1)} = \boxed{180 \text{ calendars}}$

y-coordinate of vertex $= P(180) = -(180)^2 + 360(180)$

$= 32400$ cents.

$= \boxed{\$324.00}$

Math 70

(6) Find two numbers whose sum is 60 and whose product is as large as possible.

$$\left.\begin{array}{l} A = \text{1st number} \\ B = \text{2nd number} \end{array}\right\} \quad \text{sum is } 60 \Rightarrow A+B=60$$

$$\text{product} \Rightarrow A \cdot B$$

Too many variables, and it's not a quadratic!

Solve $A+B=60$ for either variable

$$B = 60 - A$$

Substitute into product

$$\text{Product} = A(60-A)$$

distribute

$$\text{Product} = 60A - A^2$$

$$P(A) = -A^2 + 60A \qquad \text{or} \qquad f(x) = -x^2 + 60x$$

$a = -1 < 0$ has maximum.

Vertex formula

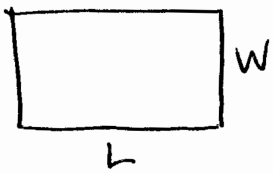
$$A = \frac{-b}{2a} = \frac{-60}{2(-1)} = 30$$

$$B = 60 - A \Longrightarrow 60 - 30 = 30$$

The two numbers are $\boxed{\text{30 and 30.}}$

Math 70

(7) The length and width of a rectangle must have a sum of 40cm. Find the dimensions of the rectangle that will have the maximum area.



Sum $= L+W = 40$

area $= L \cdot W$

Solve $L+W=40$ for a variable (doesn't matter)

$$L = 40 - W$$

substitute into area

$$\text{area} = (40-W)\cdot W$$

distribute

$$\text{area} = 40W - W^2$$

$$A(W) = -W^2 + 40W \qquad \text{or} \qquad f(x) = -x^2 + 40x$$

$a = -1 < 0$ $\cap$ maximum.

Find vertex using vertex formula

$$W = \frac{-b}{2a} = \frac{-40}{2(-1)} = \boxed{20\text{ cm} = W}$$

$$L = 40 - 20 = \boxed{20\text{ cm} = L}$$

Math 70

(8) Methane emissions $f(x) = -0.072x^2 + 1.93x + 173.9$
where $f(x)$ = amount of methane, in million metric tons
and x = # years after 2010.

a) Emissions in 2019?

$$\begin{array}{r} 2019 \\ -\ 2010 \\ \hline 9 = x \end{array}$$

Find $f(9) = -0.072(9)^2 + 1.93(9) + 173.9$

$= 185.438$ million metric tons $\approx \boxed{185.44 \text{ million metric tons}}$

b) Maximum or minimum?

$a = -0.072 < 0$ $\frown$ $\boxed{\text{maximum.}}$ $\boxed{a<0}$

c) What year at maximum? (nearest whole year)

vertex formula $\dfrac{-b}{2a} = \dfrac{-1.93}{2(-0.072)}$

$= 13.40277778$

≈ 13 years $= \boxed{2023}$

d) What is the level of methane emissions that year?
Use rounded answer from c, then round to 2 decimal places.

$f(13) = -0.072(13)^2 + 1.93(13) + 173.9$

$= 186.822$

$\approx \boxed{186.82 \text{ million metric tons}}$

Math 70

Extras

(9) 2 numbers, sum is 11, product as large as possible

$A+B=11 \Rightarrow B=11-A$

$AB=\max$

$P(A)=A(11-A)$

$=11A-A^2$

$=-A^2+11A$ or $f(x)=-x^2+11x$

vertex $\frac{-b}{2a}=\frac{-11}{2(-1)}=\frac{11}{2}=5.5$

$B=11-5.5=5.5$ $\boxed{5.5 \text{ and } 5.5}$ or $\boxed{\frac{11}{2} \text{ and } \frac{11}{2}}$

(10) difference is 10, product as small as possible.

$A-B=10 \Rightarrow A=B+10$

$A\cdot B=\min.$

$P(B)=(B+10)\cdot B$

$=B^2+10B$ or $f(x)=x^2+10x$

vertex formula $\frac{-b}{2a}=\frac{-10}{2(1)}=-5=B.$

$A=B+10 \Rightarrow A=-5+10=5$

$\boxed{5 \text{ and } -5}$

(11) difference is 8, product as small as possible.

$A-B=8 \Rightarrow A=B+8$

$AB=\min$

$P(B)=B(B+8)$

$=B^2+8B$ or $f(x)=x^2+8x$

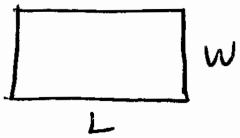
Vertex formula $\frac{-b}{2a}=\frac{-8}{2(1)}=-4=B$

$A=-4+8=4$

$\boxed{4 \text{ and } -4}$

Math 70

(12) The length and width of a rectangle must have a sum of 50. Find dimensions of rectangle with max area.



sum $= L + W = 50$
area $= L \cdot W$

Solve $L + W = 50$ for either variable

$$W = 50 - L$$

substitute

$$\text{area} = L(50 - L)$$

distribute

$$\text{area} = 50L - L^2$$

standard form

$$\text{area} = -L^2 + 50L \qquad \text{or} \qquad f(x) = -x^2 + 50x$$

vertex formula

$$\frac{-b}{2a} = \frac{-50}{2(-1)} = 25 = L$$

$$W = 50 - L \Rightarrow W = 50 - 25$$

$$\boxed{25 \times 25}$$

$\boxed{\text{CAUTION!}}$

<u>when finding vertex by GC</u>

- students who do tables get different (<u>wrong</u>) answers depending on Δx.

(13) <u>Ex.</u> $y = -16x^2 + 103x$ $\qquad y\left(\frac{103}{32}\right) = \frac{10609}{64} = 165.765625$

$$\frac{-b}{2a} = \frac{-103}{2(-16)} = 3.21875$$

$\boxed{\text{correct} \rightarrow \text{using MAX} \quad (3.21875, 165.765625)}$

<u>WRONG - USING TABLE:</u>

using $\Delta x = 1$: max $(3, 165)$ wrong!

$\Delta x = .5$ max $(3, 165)$ wrong!

$\Delta x = .1$ max $(3.2, 165.76)$ wrong!

$\Delta x = .01$ max $(3.22, 165.77)$ wrong!

$\Delta x = .001$ max (

every value from 3.213 to 3.225 rounds to 165.77

$(3.219, 165.765624)$

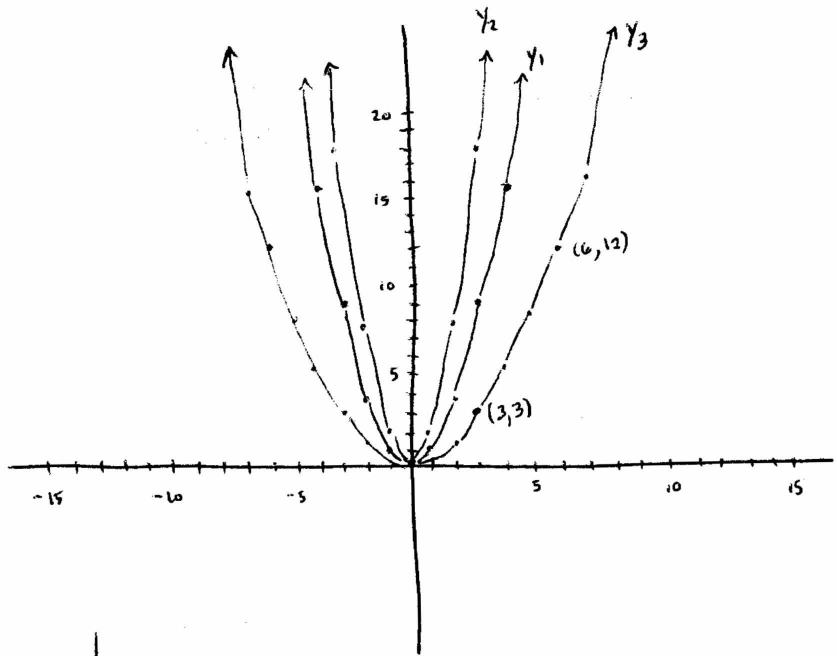
wrong!

Graph in GC and notice how the parabola's shape changes with a.
Identify each vertex.

① $y_1 = x^2$ $\boxed{V(0,0)}$

$y_2 = 2x^2$ $\boxed{V(0,0)}$

$y_3 = \frac{1}{3}x^2$ $\boxed{V(0,0)}$



$a=1$ basic shape

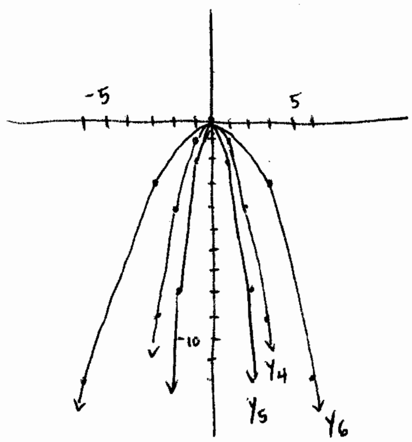
$a=2$ narrower

$a=\frac{1}{3}$ wider

② $y_4 = -x^2$ $\boxed{V(0,0)}$

$y_5 = -2x^2$ $\boxed{V(0,0)}$

$y_6 = -\frac{1}{3}x^2$ $\boxed{V(0,0)}$



$a=-1$ basic shape

$a=-2$ narrower

$a=\frac{-1}{3}$ wider

The sign of a shows direction $\begin{cases} a>0 & \text{opens up} \\ a<0 & \text{opens down} \end{cases}$

The value of a <u>without the sign</u> shows width $0<|a|<1$ wider

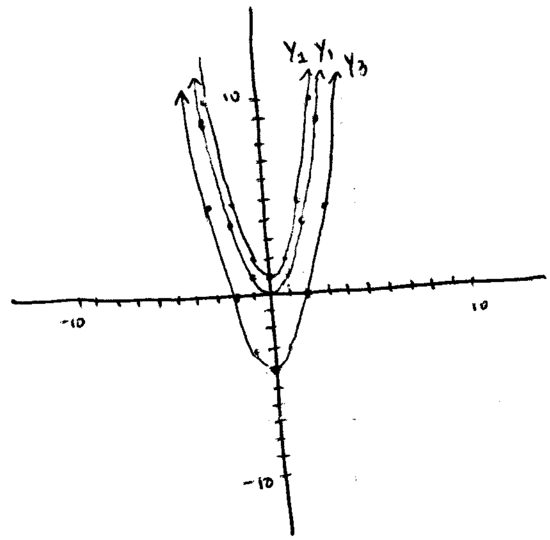
$|a|>1$ narrower.

Graph in GC and notice how the parabola's location changes. Identify each vertex.

(3) $Y_1 = x^2$ $\boxed{V(0,0)}$

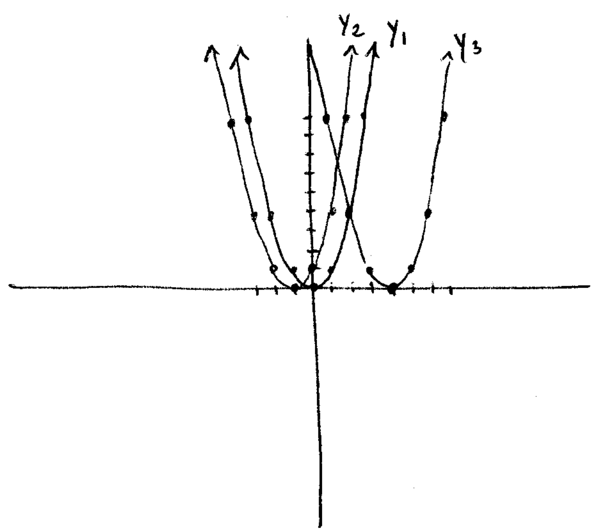
$Y_2 = x^2 + 1$ $\boxed{V(0,1)}$

$Y_3 = x^2 - 4$ $\boxed{V(0,-4)}$



Y_2 is shifted up 1 unit
(each y-coord, add 1)

Y_3 is shifted down 4 units.
(each y coord, subtract 4)



(4) $Y_1 = x^2$ $\boxed{V(0,0)}$

$Y_2 = (x+1)^2$ $\boxed{V(-1,0)}$

$Y_3 = (x-4)^2$ $\boxed{V(4,0)}$

Y_2 is shifted left 1 unit
(use opposite sign of what you see inside the parentheses)

Y_3 is shifted right 4 units

$f(x) = ax^2 + k$ shift k units vertically using sign of k (+) up (−) down

$f(x) = a(x-h)^2$ shift −h units horizontally (using opposite of sign of h) (−) right (+) left

Without graphing, identify the vertex, direction and width relative to $y=x^2$.

5. $f(x) = (x-2)^2 + 3$

VERTEX FORM OF EQUATION OF A PARABOLA

$y = a(x-h)^2 + k$

vertex (h,k)

$\boxed{V(2,3)}$

$a=1$ $\boxed{\text{opens up}}$ $a>0$

$\boxed{\text{Same width as } y=x^2}$ $|a|=1$

6. $f(x) = -\frac{1}{2}(x+4)^2 - 5$

$\boxed{V(-4,-5)}$

$a = -\frac{1}{2}$ $\boxed{\text{opens down}}$ $a<0$

$\boxed{\text{wider than } y=x^2}$ $0<|a|<1$

7. $f(x) = 3(x-1)^2 - 6$

$V(1,-6)$

$a=3$ $\boxed{\text{opens up}}$ $a>0$

$\boxed{\text{narrower than } y=x^2}$ $|a|>1$

8. $f(x) = -4(x+2)^2 - 1$

$\boxed{V(-2,-1)}$

$a = -3$ $\boxed{\text{opens down}}$ $a<0$

$\boxed{\text{narrower than } y=x^2}$ $0<|a|<1$

9. $f(x) = -(x-7)^2 + 2$

$\boxed{V(7,2)}$

$a=-1$ $\boxed{\text{opens down}}$ $a<0$

$\boxed{\text{Same width as } y=x^2}$ $|a|=1$

So how do we go from $f(x) = ax^2+bx+c$ to $f(x) = a(x-h)^2+k$??

10 $f(x) = 2(x-1)^2+3$

a) Find vertex using vertex form $y = a(x-h)^2+k$

$\boxed{V(1,3)}$

b) Simplify $f(x) = 2(x-1)^2+3$.

FOIL $f(x) = 2(x-1)(x-1)+3$

$f(x) = 2(x^2-2x+1)+3$

$f(x) = 2x^2-4x+2+3$

$\boxed{f(x) = 2x^2-4x+5}$

c) Find the vertex of $f(x) = 2x^2-4x+5$ using vertex formula.

$h = \frac{-b}{2a} = -\frac{(-4)}{2(2)} = \frac{+4}{4} = 1$

$k = f(1) = 2(1)^2-4(1)+5$

$= 2-4+5$

$= 3$

$\boxed{V(1,3)}$ whew! It should be the same!

Notice the algebra in b), we want to do this backward

Forward:

$2(x-1)^2+3$

$= 2(x-1)(x-1)+3$

$= 2(x^2-2x+1)+3$

$= 2x^2-4x+2+3$

$= 2x^2-4x+5$

Backward:

$2x^2-4x+5$

These two steps are key.

$= 2x^2-4x+\underbrace{2+3}$

why $2+3$? not $1+4$ or $4+1$ or $3+2$?? or $6-1$?

$= 2(x^2-\ x+1)+3$

factor GCF from 3 terms

$= 2(x-1)(x-1)+3$

$= 2(x-1)^2+3$

factor perfect square